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Effect of linear coupling on nonlinear resonances in betatron motion

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The influence of linear coupling on nonlinear resonances in betatron motion is considered. A model of lattice with a single sextupole and a linearly coupled one-turn matrix is analyzed. The perturbative approach based on normal forms is considered, and the relation of the first resonant coefficient of the interpolating Hamiltonian with the island width or with the unstable separatrices is outlined. The dependence of the first resonant coefficient on the coupling angle is worked out for generic resonances. The analytical results are in very good agreement with the numerical simulations based on tracking and frequency analysis. [S1063-651X(97)03302-3]

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I. INTRODUCTION

In circular accelerators there are several sources of linear coupling between the vertical and the horizontal betatron oscillations. The longitudinal magnetic field of solenoids creates linear coupling that can be locally corrected by skew quadrupoles; moreover, superconducting dipoles, which are used both in present and in future large hadron colliders [1], do contain unwanted skew quadrupole components; other sources of linear coupling are the misalignments of the magnets.

A theory of linear coupling has been developed several years ago [2,3]; one can define a symplectic rotation that transforms a generic linearly coupled matrix to an uncoupled one. Such transformation depends on the coupling angle φ that measures the deviation from the uncoupled case. The effect of linear coupling on the linear optics of the lattice (i.e., beta functions, linear tunes, and Courant-Snyder invariants) has been worked out during the last decades [4,5]. Moreover, clever ways to decouple the linear motion using some families of correcting skew quadrupoles have been defined (see, for instance, [6,7]). On the other hand, much less work has been developed to analyze the influence of linear coupling on the nonlinear motion [5,8], and on the dynamic aperture.

Unfortunately, no analytical tools are available for a direct evaluation of the dynamic aperture (with the exception of the case of the working point close to unstable low-order resonances). On the other hand, the perturbative formalism based either on Hamiltonian flows [9,10,14] or directly on the one-turn map [11–13,15,16] provides a lot of relevant information on nonlinear resonances.

In this paper we outline the results obtained in Ref. [17], where the effect of linear coupling on nonlinear resonances is analyzed for a lattice model made up of a normal sextupole and a linearly coupled one-turn matrix. Using the perturbative tools of normal forms [13,15,18], we compute the value of the first resonant coefficient (i.e., the *leading term*) in the interpolating Hamiltonian for several resonances and for different coupling angles. Order-independent codes are used to automatically evaluate the normal form series of a generic one-turn map (see Refs. [19,20]). Then, the scaling laws that relate the island width (stable resonances) or the dynamic aperture (unstable resonances) to the leading term of the resonance are outlined, and the dependence of the leading term on the coupling angle φ is worked out. One finds that in the uncoupled case some resonances have zero leading terms, and that for weak linear coupling all these coefficients grow proportionally to φ. At the same time, all the other resonances have a leading term that decreases proportionally to φ^2. In order to check out this analytical result, we use a numerical method to visualize the network of resonances and their widths using short-term tracking and frequency analysis [21–24]; the numerical simulations fully confirm the analytical results.

II. MODEL: HÉNON MAP WITH LINEAR COUPLING

We consider a lattice made up of a linear part plus a sextupole of unitary strength in the one-kick approximation [13], whose one-turn map reads

$$\begin{pmatrix} x' \\ p_x' \\ y' \\ p_y' \end{pmatrix} = \mathbf{L} \begin{pmatrix} x \\ p_x + (x^2 - y^2) \\ y \\ p_y - 2xy \end{pmatrix}. \tag{1}$$

The transfer matrix L can be factorized according to the well-known formula [2]

$$L = \begin{pmatrix} Ic & D^{-1}s \\ -Ds & Ic \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} Ic & -D^{-1}s \\ Ds & Ic \end{pmatrix}, \tag{2}$$

where A, B, D, and I are two-dimensional (2D) matrices (I is the identity), $s = \sin\varphi$, $c = \cos\varphi$, and φ is the coupling angle. Even though not all the transfer matrices can be written in the above form, this is sufficiently generic for our purposes. The matrices A and B can be written in the Courant-Snyder form [25], and therefore one can define the linear transformation T and the new coordinates (u, p_u, v, p_v)

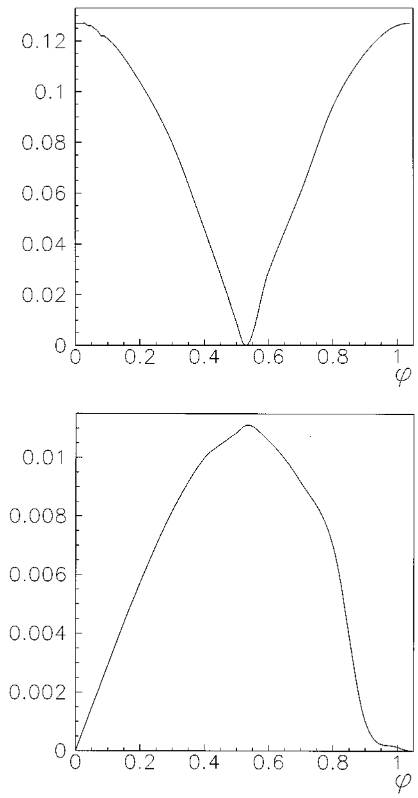

FIG. 1. Dependence of the leading term $h_{0,0,1}$ of resonance (1,2) (top) and (4,1) (bottom) on the coupling angle φ for a 4D Hénon map linearly coupled.

$$\begin{pmatrix} u \\ p_u \\ v \\ p_v \end{pmatrix} = \mathbf{T}^{-1} \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{3}$$

that reduce the linear part of the map to the direct product of two 2D rotations of angles ω_1 and ω_2

$$\mathbf{T}^{-1}\mathbf{L}\mathbf{T} = \begin{pmatrix} R(\omega_1) & 0 \\ 0 & R(\omega_2) \end{pmatrix}. \tag{4}$$

The map expressed in the coordinates (u,p_u,v,p_v) will be used for numerical simulations. In order to apply the normal form theory, it is customary to diagonalize the linear part of the motion, i.e., switch to the complex coordinates $z_1 = u - ip_u$ and $z_2 = v - ip_v$. In these variables, the map reads

$$z_1' = e^{i\omega_1} z_1 + F_1(z_1, z_1^*, z_2, z_2^*)$$

$$z_2' = e^{i\omega_2} z_2 + F_2(z_1, z_1^*, z_2, z_2^*), \tag{5}$$

where F_1 and F_2 are polynomials of second degree in the variables (z_1, z_1^*, z_2, z_2^*), whose coefficients depend on the linear parameters φ, A, B and D. For $\varphi = 0$ most of the monomial coefficients are zero, and one obtains the four-dimensional (4D) Hénon map [13].

In simulations we set the matrix D to the identity: one can show [17] that this is a rather good approximation for linear coupling generated by skew quadrupoles or solenoids. Moreover, we assume that the β functions in the sextupole have the same value and that their derivatives are equal to zero. The fractional parts of the working point are fixed on the proposed LHC value: $\omega_1/2\pi = 0.28$ and $\omega_2/2\pi = 0.31$.

III. NONLINEAR RESONANCES THROUGH NORMAL FORMS

Once the linear part of the map is diagonalized, one can apply the standard normal form procedure to extract the resonance parameters. We will only outline the procedure, and we refer to the existing literature [11–13,18,20] for a more complete description. The nonlinear map F is transformed through a nonlinear conjugating function to the normal form U, whose polynomial structure is much simpler (i.e., most of its monomials are zero). The normal form U is then expressed as the Lie series of an interpolating Hamiltonian. Let $(\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*)$ be the normal form coordinates, and let $(\theta_1, \theta_2, \rho_1, \rho_2)$ be the amplitude-angle variables that are more suitable for our analysis: one has $\zeta_j = \sqrt{\rho_j}\exp(i\theta_j)$ for $j = 1,2$. The Hamiltonian for a resonance (q,p) in these variables reads

$$h = \sum h_{k_1,k_2,l} \;(\rho_1)^{k_1 + lq/2} (\rho_2)^{k_2 + l|p|/2} \times \cos[l(q\theta_1 + p\theta_2) + \varphi_{k_1,k_2,l}], \tag{6}$$

where $q \in \mathbf{N}$ and $p \in \mathbf{Z}$ denote the resonance. Since h contains only one combination of angles, then there is a second invariant $r_2 = p\rho_1 - q\rho_2$ and h can be reduced to a 2D pendulum Hamiltonian with a parametric dependence on r_2.

In this paper we will focus on the coefficient $h_{0,0,1}$: this is the first resonant coefficient, i.e., the coefficient of the monomial in the Hamiltonian that depends on the angles and whose order in the amplitudes is minimum; $h_{0,0,1}$ can be equivalently called the *leading term* of the resonance. We consider the generic case of an accelerator whose first-order detuning is different from zero: one can prove [13] that if the resonance order $q + |p|$ is greater than four, then the reduced Hamiltonian features a chain of islands whose area Σ scales with $h_{0,0,1}$ according to

$$\Sigma \propto \sqrt{h_{0,0,1}}. \tag{7}$$

The total hypervolume in the 4D phase space of initial conditions that are locked on the resonance can be obtained by integrating the island area along the second invariant r_2, and, therefore, it has the same dependence on the leading term. In the case of resonances of order 3, if $h_{0,0,1} \neq 0$ then the reduced Hamiltonian features a separatrix that limits the stability domain, whose distance D to the origin scales according to

$$D \propto (h_{0,0,1})^{-2}. \tag{8}$$

In the case of resonances of order 4 one can have both situations. If the leading term is dominant over the first-order detuning terms, there is a separatrix whose distance to the origin scales according to

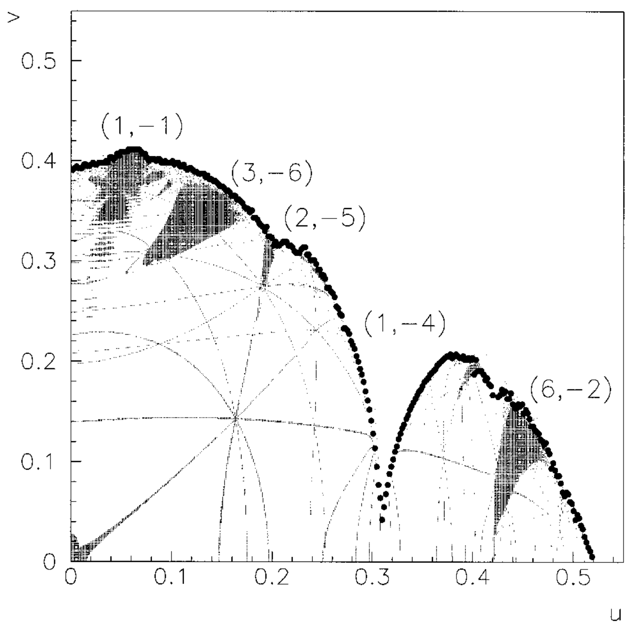

FIG. 2. Network of resonances for a 4D Hénon map without linear coupling; some resonances are indicated above their channels.

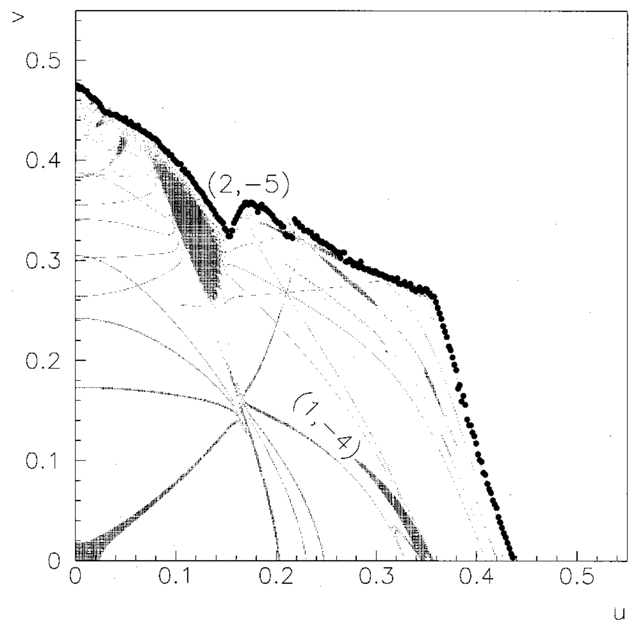

FIG. 4. Network of resonances for a 4D Hénon map with linear coupling $\varphi=\pi/6$; some resonances are indicated above their channels.

$$D\propto(h_{0,0,1})^{-1}. \tag{9}$$

Otherwise, there is an island whose area has the same dependence of Eq. (7) on the leading term.

IV. NORMAL FORM RESULTS

Using the code ARES [20], we evaluated the leading terms of several resonances for a model without linear coupling [$\varphi=0$ in Eq. (2)] and with a very weak linear coupling. The results show that, if p is even, a resonance (q,p) has a nonzero leading term for $\varphi=0$; if p is odd, then the leading term for $\varphi=0$ is zero. This property is due to the absence of some monomials in the quadratic nonlinearity, i.e., to the fact that the sextupole has no skew component. Indeed, numerical simulations show that a weak linear coupling generates nonzero leading terms for all the resonances (q,p) with odd p.

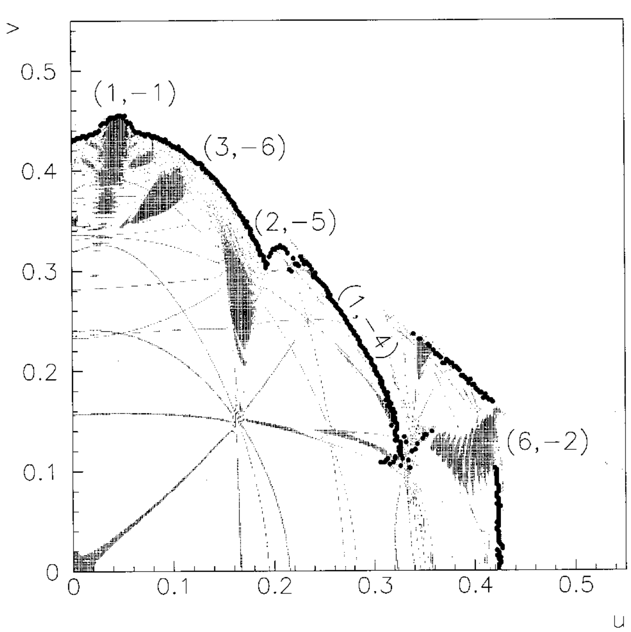

FIG. 3. Network of resonances for a 4D Hénon map with linear coupling $\varphi=\pi/12$; some resonances are indicated above their channels.

We have also analyzed the dependence of the leading term on the coupling angle φ. In Fig. 1, top, we plot the case of resonance (1,2), that features a nonzero leading term for $\varphi=0$. In Fig. 1, bottom, the same plot is given for resonance (4,1), that has a zero leading term for $\varphi=0$. It turns out that $h_{0,0,1}$ has the following dependence on φ: for resonances with zero leading term at $\varphi=0$ one has

$$h_{0,0,1}(\varphi)=\kappa_1\varphi+O(\varphi^2) \tag{10}$$

whilst in the other case one has

$$h_{0,0,1}(\varphi)=\kappa_0-\kappa_2\varphi^2+O(\varphi^3) \tag{11}$$

where κ_0,κ_1 and κ_2 are positive constants. This behavior has been verified for several cases; an analytical proof, based on the algebraic evaluation of the dependence of $h_{0,0,1}$ on ϕ, has been given for resonances of order 3 and 4 [17].

The main result of this analysis is that the linear coupling increases the strength of resonances with zero leading term in the uncoupled case, and decreases the effect of the other ones. Usually, the maximum in $h_{0,0,1}(\varphi)$ for one type of resonances corresponds to the minimum in $h_{0,0,1}(\varphi)$ for the other ones. This result will be confirmed by numerical simulations in Sec. V.

V. NUMERICAL RESULTS

In order to check out the analytical results, we use a method to visualize the network of resonances and their strength in phase space, using short-term tracking and frequency analysis [21–24]. We consider a very dense grid of initial conditions (400×400) in the plane (u,v), and we set the momenta (p_u, p_v) to zero. For each initial condition we start a short-term tracking (1024 turns) and we compute the frequencies of the orbit using the interpolation of the fast Fourier transform (FFT) plus Hanning filter [23]. If the nonlinear frequencies (ν_1, ν_2) satisfy a resonant condition

$$q\nu_1 + p\nu_2 = l + \epsilon \quad \epsilon \ll 1 \tag{12}$$

with $q \in \mathbb{N}$, $p,l \in \mathbb{Z}$, then the initial condition is locked on the resonance (q,p). The plot in the plane (u,v) of only these initial conditions provides a picture of the network of resonances: large resonant channels correspond to strong resonances, and vice versa.

In Fig. 2 we give this plot for the map (1) without linear coupling ($\varphi = 0$); large dots represent the short-term dynamic aperture. One can see that the largest channels correspond to resonances $(6,-2)$, $(3,-6)$, and $(1,-1)$, that have a nonzero leading term. Resonance $(1,-4)$ as well has a very strong leading term: there is no phase locking and the resonance splits the stability domain. The channel of resonance $(2,-5)$, which has a zero leading term, is rather small. In Figs. 3 and 4 we plot the same picture for the same model with $\varphi = \pi/12$ and $\varphi = \pi/6$, respectively: in Fig. 3 resonances $(6,-2)$, $(3,-6)$, and $(1,-1)$ are less strong and in Fig. 4 they disappear, whilst resonance $(2,-5)$ becomes more and more relevant. This agrees with the perturbative analysis carried out through normal forms. In order to better visualize the mechanism of resonance excitation and deexcitation we have considered extremely large coupling angles.

Finally, we remark that in presence of strong linear coupling a new phenomenon appears on resonance crossings: whilst for $\varphi = 0$ resonances belonging to the same resonance nest cross in the same point (see Fig. 2, at $u \approx 0.16$ and $v \approx 0.14$), for large values of the coupling angles one has a splitting of the crossing (see Fig. 4, same place); this is typical of systems with strong linear coupling, and it has been observed also in other fields of physics [26].

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